Secure Domination in Bisplit Graphs - An Algorithmic Study

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Abstract. A dominating set S of a graph $G = (V, E)$ is called a *secure dominating set* if each vertex $u \in V(G) \setminus S$ is adjacent to a vertex $v \in S$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The *Minimum Secure Domination problem* (MSD) is to find a secure dominating set of a graph G of minimum cardinality. In this paper, the computational complexity of the secure domination problem on several graph classes is investigated. The decision version of secure domination problem was shown to be NP-complete on bisplit graphs. We further focus on complexity analysis of secure domination problem under additional structural restrictions on bisplit graphs. In particular, by imposing chordality as a parameter, we establish the P versus NPC dichotomy status of secure domination problem within bisplit graphs. In addition, we show MSD is polynomial time solvable in chain graphs. We also prove that MSD cannot be approximated for a bisplit graph within a factor of $(1 - \epsilon)\ ln\ |V|$ for any $\epsilon > 0$, unless $NP \subseteq DTIME(|V|^{O(log\ log\ |V|)})$. On the positive side, we show that MSD can be approximated within a factor of $O(ln\ |V|)$ for any graph G with $\delta(G) \geq 2$.

Keywords: Secure domination · Bisplit graphs · Chordality · Chain graph · Approximation algorithms · Inapproximability

1 Introduction

Domination and its variants in graph theory have been extensively studied due to their wide range of applications in areas such as computer networks, social networks, and locational studies. A practical scenario arises in securing a spatial domain such as a military terrain, university campus, or residential complex where guards must be positioned in such a way that every location is either directly guarded or can be reached by a guard from a neighbouring location, without compromising on overall security. This setup can be modelled using graphs, where vertices represent locations and edges represent accessible paths for guard movement. The challenge is to identify a set of vertices such that every vertex is either in the set or adjacent to a vertex in the set, forming a dominating set. Additionally, for every vertex not in the set, there must be a neighbouring

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vertex in the set such that swapping their positions still results in a dominating set. The objective is to find such a minimum set of vertices. This precisely defines the *secure domination problem* [1].
Let $G = (V, E)$ be a graph. A subset $D \subseteq V(G)$ is a *dominating set* of G if every vertex in $V(G) \setminus D$ has at least one neighbour in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The *domination problem* is to find a minimum dominating set of a graph. A variation of domination, *secure domination*, was introduced by Cockayne *et al.*[2]. A dominating set S of G is called a *secure dominating set* if each vertex $u \in V(G) \setminus S$ is adjacent to a vertex $v \in S$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The *secure domination number* $\gamma_s(G)$ of G is the minimum cardinality of a secure dominating set of G. The *Minimum Secure Domination problem* MSD is to find a secure dominating set of a graph G of cardinality $\gamma_s(G)$. Let DD denote the decision version of the domination problem and let SDD denote the decision version of the secure domination problem.
Domination Problem *(DD):*
Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
Question: Does G have a dominating set of cardinality at most k?
Secure Domination Problem *(SDD):*
Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
Question: Does G have a secure dominating set of cardinality at most k?
The problem of secure domination was introduced by *Cockayne et al.*, who investigated some fundamental properties of a secure dominating set, and also obtained exact values of $\gamma_s(G)$ for some graph classes, such as paths, cycles and complete multipartite graphs[2]. Various properties and characterisations of secure domination set have been researched [3,4,5,6]. Upper and lower bounds on $\gamma_s(G)$ have been established for some graph classes [4,5,6,7,8]. The problem of computing secure domination number is investigated for some restricted graph classes and their complexity status is found. The problem is NP-complete in general, and remains NP-complete when restricted to bipartite and split graphs [6], star convex bipartite graphs and doubly chordal graphs [9], and chordal bipartite graphs and undirected path graphs [10]. On the positive side, the problem is linear-time solvable on trees (subclass of bipartite) [1]. On subclasses of chordal graphs such as block graphs [10], and proper interval graphs [11,12] the problem is linear as well. The problem is linear in cographs[13,14]. In addition, the problem is APX-complete for graphs with maximum degree 4 and there exists an inapproximability result for the problem [9].
SDD is found to be NP-complete on split graphs and bipartite graphs, a natural direction of research is to investigate the computational complexity of the problem on *bisplit graph*, a bipartite analog of a split graph. In Section 3 we show that SDD on bisplit graphs is NP-complete. An interesting dichotomy is obtained by imposing chordality as a structural parameter on bisplit graphs. In Section 3.1, we provide an algorithm for computing the secure domination of a *chordal bisplit graph*. In Section 3.2, we show that SDD is NP-complete on *chordal bipartite bisplit graphs*. In Section 3.3 we establish MSD is polynomial

time solvable in chain graphs. Section 4 establishes hardness and approximation results for secure domination problem.

2 Preliminaries

In this section, we give some pertinent definitions and state some preliminary results. Let $G = (V, E)$ be a finite, simple, undirected and connected graph. Given $S \subseteq V$ and $v \in S$, a vertex $u \in V \setminus S$ is an S-external private neighbour (S-epn) of v if $N(u) \cap S = \{v\}$. The set of all S-epn of v is denoted by $epn(v, S)$. Some other notation and terminology not introduced here can be found in [15]. A *split graph* is a graph which can be partitioned into a clique and an independent set. An undirected graph $G = (X, Y, Z, E)$ is a *bisplit graph* if its vertex set can be partitioned into three stable sets X, Y and Z such that $Y \cup Z$ induces a complete bipartite subgraph (a bi-clique) in G. A bipartite graph $G = (X, Y, E)$ is a chain graph, if there exists a chain ordering of $X \cup Y$, i.e. $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ such that $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_n)$ and $N(y_1) \supseteq N(y_2) \supseteq \cdots \supseteq N(y_m)$.

3 Secure Domination in Bisplit graphs

Proposition 1 *[2] If D is a secure dominating set of a graph G, then for every vertex $v \in D$, the subgraph induced by $epn(v, D)$ is complete.*

Theorem 1. *SDD is NP-complete for bisplit graphs.*

Proof. *SDD is in NP* : Given a graph $G = (V, E)$ and a certificate $D \subseteq V(G)$, we show that there exists a deterministic polynomial time algorithm for verifying the validity of D. Note that it is easy to check whether $|D| \leq k$. For each vertex in $V \setminus D$ we may scan through its adjacency list and ensure it has a vertex of D this takes at most n^2 comparisons, where n is the cardinality of set V. Now for each vertex in $V \setminus D$ there are at most k neighbours in D. So to verify whether the swap set is dominating takes at most n^3 steps. The certificate verification can be done in $O(n^3)$. Thus, we conclude that SDD is in NP.
It is known that DD in bisplit graphs is NP-complete [16] and this can be reduced to SDD in bisplit graphs using the following reduction.
Construction: Let $G = X \cup Y \cup Z$ be a bisplit graph whose vertex set is partitioned into three stable sets X, Y and Z such that $Y \cup Z$ induces a complete bipartite graph. Let G^* be obtained from G as follows. We add a path $P_4 : x - y - z - x'$ such that, x is adjacent to y; y is adjacent to x, z, and all vertices of X and Z; z is adjacent to y, x', and all vertices of Y and x' is adjacent to z. We note that $|V(G^*)| = |V(G)| + 4$ and $|E(G^*)| = |E(G)| + |V(G)| + 3$. Thus G^* can be constructed from G in polynomial time.

Claim 3.1. G^* is a bisplit graph.

Proof. Let $G^* = X^* \cup Y^* \cup Z^*$ where $X \subseteq X^*$, $Y \subseteq Y^*$ and $Z \subseteq Z^*$. We show that G^* is a bisplit graph where X^*, Y^* and Z^* are three stable sets and $Y^* \cup Z^*$

forms a biclique. By our construction, y is adjacent to all vertices of X and Z. So $y \in Y^*$. We know that $Y \cup Z$ of G forms a biclique. x and x' are neither adjacent to all vertices of Y nor adjacent to all vertices of Z. These two vertices cannot be a part of biclique. $\{x, x'\} \in X^*$. Note that z is adjacent to $y \in Y^*$, $x' \in X^*$ and all vertices of Y, which implies $z \in Z^*$. Therefore, the graph $G^* = X^* \cup Y^* \cup Z^*$ where $X^* = X \cup \{x, x'\}$, $Y^* = Y \cup \{y\}$ and $Z^* = Z \cup \{z\}$. Note that X^*, Y^* and Z^* are three stable sets and $Y^* \cup Z^*$ forms a biclique. Hence G^* is a bisplit graph.

Claim 3.2. G has a dominating set D with $|D| \leq k$ if and only if G^* has a secure dominating set D^* with $|D^*| \leq k^* = k + 2$.

Proof. Let G have a domination set D of size at most k. We shall prove that G^* has a secure dominating set D^* with $|D^*| \leq k^* = k+2$. Consider $D^* = D \cup \{y, z\}$. Clearly D^* is a dominating set of G^*. We wish to prove for each vertex $v \in V(G^*) \setminus D^*$, there exists a neighbour $v^* \in D^*$ such that *swap set* $(D^* \setminus \{v^*\}) \cup \{v\}$ is again a dominating set of G^*. When $v \in V(G) \setminus D$, since D is a subset of D^* and D is a dominating set of G, for every vertex $v \in V(G) \setminus D$, there exists a neighbour $v' \in D \subseteq D^*$. Therefore, for each $v \in V(G) \setminus D$, there exists a neighbour $v' \in D^*$ such that $(D^* \setminus \{v'\}) \cup \{v\}$ is a dominating set of G^*. When $v = x$ (x') there exists a neighbour $y \in D^* (z \in D^*)$ such that $(D^* \setminus \{y\}) \cup \{x\}$ $((D^* \setminus \{z\}) \cup \{x'\})$ is a dominating set of G^*. Therefore, $D^* = D \cup \{y, z\}$ is a secure dominating set of G^* with $|D^*| \leq k^* = k + 2$. Conversely, let G^* have a secure dominating set D^* with $|D^*| \leq k^* = k + 2$. We consider the following cases depending on the value of $|D^* \cap \{y, z\}|$ and we shall prove that G has a dominating set D with $|D| \leq k$ in each of these cases.
Consider $D' = D^* \cap V(G)$. If D' dominates G, then $D = D'$ and we are done. So let us assume that D' is not a dominating set of G. Let W be the non-empty set of vertices of G having no neighbour in D'. Let $W_x = W \cap X$, $W_y = W \cap Y$ and $W_z = W \cap Z$. Since W is nonempty, at least one of the sets W_x, W_y or W_z is nonempty.
Case 1. $|D^* \cap \{y, z\}| = 2$
Case 1.1. $|D^* \cap \{x, x'\}| = 2$. Since D' does not dominate the vertices of $W \subseteq V(G) \setminus D^*$, each vertex of $W_x \cup W_z$ is dominated by $y \in G^*$ and each vertex of W_y is dominated by $z \in G^*$. By Proposition 1, if $W_x \cup W_z \neq \phi$ then, $w \in W_x \cup W_z$ is sufficient to dominate it and if $W_y \neq \phi$ then $|W_y| = 1$. Therefore $D = D' \cup \{w\} \cup W_y$ will be the dominating set of G and we see that $|D| = |D^*| - |D^* \cap \{y, z\}| - |D^* \cap \{x, x'\}| + 2 \leq k^* - 2 - 2 + 2 = k$.
Case 1.2. $|D^* \cap \{x, x'\}| = 1$. Here $|D'| = |D^*| - |D^* \cap \{y, z\}| - |D^* \cap \{x, x'\}| \leq k^* - 2 - 1 = k - 1$. By our assumption since D' does not dominate the vertices of $W \subseteq V(G) \setminus D^*$, we construct D as follows. If $x \in D^*$ by our case $x' \notin D^*$. Since D^* is a secure domination set of G^*, $(D^* \setminus \{z\}) \cup \{x'\}$ is a dominating set of G^*, which implies all vertices of Y are dominated by $D' \subset D^*$ and $W = W_x \cup W_z \neq \phi$. By Proposition 1, $epn(y, D^*)$ is complete, so $w \in W_x \cup W_z$ is sufficient to dominate it. Therefore $D = D' \cup \{w\}$ will be the dominating set of G and $|D| = |D'| + 1 \leq k$. If $x' \in D^*$ by our case $x \notin D^*$. Since D^* is a

secure domination set of G^*, $(D^* \setminus \{y\}) \cup \{x\}$ is a dominating set of G^*, which implies all vertices of X and Z are dominated by $D' \subset D^*$ and $W = W_y \neq \phi$. By Proposition 1, $epn(z, D^*)$ is complete, so $|W_y| = 1$. Therefore $D = D' \cup W_y$ will be the dominating set of G and $|D| = |D'| + 1 \leq k$.
Case 1.3. $|D^* \cap \{x, x'\}| = 0$. Since $x \notin D^*$ and D^* is a secure domination set of G^*, $(D^* \setminus \{y\}) \cup \{x\}$ is a dominating set of G^*, which implies all vertices of X and Z are dominated by $D' \subset D^*$. Additionally, $x' \notin D^*$ and since D^* is a secure domination set of G^*, $(D^* \setminus \{z\}) \cup \{x'\}$ is a dominating set of G^*, which implies all vertices of Y are dominated by $D' \subset D^*$. Therefore, our assumption fails and consequently $D = D'$ forms a dominating set of G and $|D| = |D'| = |D^*| - |D^* \cap \{y, z\}| - |D^* \cap \{x, x'\}| \leq k^* - 2 + 0 = k$.
Case 2. $|D^* \cap \{y, z\}| = 1$
Case 2.1. $y \in D^*$. By our case $z \notin D^*$, which implies $x' \in D^*$ securely dominates $\{x', z\}$ and all vertices of Y are dominated by $D' \subset D^*$. So, $W = W_x \cup W_z$ and we construct D as follows. If $x \notin D^*$, since D^* is a secure domination set of G^*, $(D^* \setminus \{y\}) \cup \{x\}$ is a dominating set of G^*, which implies all vertices of X and Z are dominated by $D' \subset D^*$. Therefore, our assumption fails and consequently $D = D'$ forms a dominating set of G and $|D| = |D'| = |D^*| - |D^* \cap \{y, z\}| - |D^* \cap \{x, x'\}| \leq k^* - 1 - 1 = k$. If $x \in D^*$ by our case $y \in D^*$, we conclude $epn(y, D^*) = W$. By Proposition 1, $epn(y, D^*)$ is complete, so $w \in W$ is sufficient to dominate it. Therefore $D = D' \cup \{w\}$ will be the dominating set of G and $|D| = |D'| + 1 = |D^*| - |D^* \cap \{y, z\}| - |D^* \cap \{x, x'\}| + 1 \leq k^* - 1 - 2 + 1 = k$.
Case 2.2. $z \in D^*$. By our case $y \notin D^*$, which implies $x \in D^*$ securely dominates $\{x, y\}$ and all vertices of X and Z are dominated by $D' \subset D^*$. So, $W = W_y$ and we construct D as follows. If $x' \notin D^*$, since D^* is a secure domination set of G^*, $(D^* \setminus \{z\}) \cup \{x'\}$ is a dominating set of G^*, which implies all vertices of Y are dominated by $D' \subset D^*$. Therefore, our assumption fails and $D = D'$ forms a dominating set of G and $|D| = |D'| = |D^*| - |D^* \cap \{y, z\}| - |D^* \cap \{x, x'\}| \leq k^* - 1 - 1 = k$. If $x' \in D^*$ by our case $z \in D^*$, we conclude $epn(z, D^*) = W$. By Proposition 1, $epn(z, D^*)$ is complete, so $|W_y| = 1$. Therefore $D = D' \cup W_y$ will be the dominating set of G and $|D| = |D'| + 1 = |D^*| - |D^* \cap \{y, z\}| - |D^* \cap \{x, x'\}| + 1 \leq k^* - 1 - 2 + 1 = k$.
Case 3. $|D^* \cap \{y, z\}| = 0$. Then, $|D^* \cap \{x, x'\}| = 2$. That is x and x' securely dominates $\{y, x\}$ and $\{z, x'\}$ respectively, which implies all vertices of G are dominated by $D' \subset D^*$. Therefore, our assumption fails and $D = D'$ forms a dominating set of G and $|D| = |D'| = |D^*| - |D^* \cap \{y, z\}| - |D^* \cap \{x, x'\}| \leq k^* - 0 - 2 = k$.
Hence in all the above cases G has a dominating set D with $|D| \leq k$.
Thus from Claims 3.1 and 3.2, Theorem 1 follows.

A deeper inspection of Theorem 1 reveals that the computational hardness of *SDD* on bisplit graphs stems from the presence of cycles in the input graphs. This observation naturally motivates the following questions. First, what is the computational complexity of *SDD* on bisplit graphs when the length of induced cycles is bounded—for instance, on chordal bisplit, chordal bipartite bisplit, and other special bisplit graphs? Second, can the maximum cycle length serve as a decisive structural parameter that dictates the polynomial time versus NP-complete dichotomy of SDD on bisplit graphs?

3.1 Chordal Bisplit graphs

Theorem 2. *[16]A graph $G = X \cup Y \cup Z$ is a chordal bisplit graph, if and only if the following properties are satisfied.*
1. The biclique $Y \cup Z$ in G is $K_{1,l}$ for some $l > 0$.
2. For each vertex $x \in X$, if $d_G(x) \geq 2$, then x is adjacent to y_1.
3. The graph induced on $Z \cup X$ is a forest.

Using Theorem 2 we show that MSD is polynomial time solvable on chordal bisplit graphs.
Let the vertices of the bisplit graph $G = (X, Y, Z, E)$ be labelled as $Y = \{y_1\}$; $Z = \{z_1, z_2, \ldots, z_l \mid l \geq 1\}$ where $Y \cup Z$ induces a biclique and $X = \{x_1, x_2, \ldots, x_t \mid t \geq 1\}$ forms an independent set. Without loss of generality let all the pendant vertices adjacent to y_1 belong to the set $Z' \subseteq Z$. If $Z' \neq \phi$ then let $z = z_i (1 \leq i \leq l)$ be a vertex in Z'. Let the other pendant vertices of G that are non adjacent to y_1 belong to the set $X' \subseteq X$. Define $Z'' = \{z_i \mid d_{X'}(z_i) = n; n \geq 2, 1 \leq i \leq l\}$. For each vertex $v \in Z''$, define $S_v = N_{X'}(v) \setminus \{x_k\}$, where $x_k (1 \leq k \leq t)$ is a vertex in set $N_{X'}(v)$. Let $X'' = \bigcup\limits_{v \in Z'} S_v$.
Consider $G' = G \setminus (Z' \cup X'' \cup Y)$. We observe G' is a forest by Theorem 2. The *minimum dominating set* of G' can be obtained in linear-time by applying known algorithms for trees [17]. Let D denote the minimum dominating set of G'. Further, let G'' be a sub forest of G' obtained by selecting some of its connected components. The selected components are precisely: (i) components isomorphic to P_2, where each vertex has *degree* ≥ 2 in G; (ii) components isomorphic to P_4; and (iii) components isomorphic to P_7. Let C denote any component in G'.

Lemma 1. *For $n \geq 8$, $\gamma(P_n) < \gamma_s(P_n)$.*

Proof. Let us prove this by induction on n. It is known from *Cockayne et al.*[2] that, $\gamma(P_n) = \lceil n/3 \rceil$ and $\gamma_s(P_n) = \lceil 3n/7 \rceil$. For $n = 8$, $\gamma(P_8) = 3$ and $\gamma_s(P_8) = 4$. For $n \geq 8$, assume that $\gamma(P_n) < \gamma_s(P_n)$. Consider $\gamma(P_{n+1})$, $\lceil \frac{n+1}{3} \rceil \leq \lceil \frac{n}{3} \rceil + 1$. By the induction hypothesis, $\lceil \frac{n+1}{3} \rceil \leq \lceil \frac{3n}{7} \rceil + 1 \leq \lceil \frac{3(n+1)}{7} \rceil$. Therefore, $\gamma(P_n) < \gamma_s(P_n)$ is true for all $n \geq 8$.

Proposition 2 *[2] For the complete bipartite graph $K_{p,q}$ where $p \leq q$,*

$$(a) \quad \gamma = \begin{cases} 1, & p = 1, \\ 2, & p > 1. \end{cases} \qquad (b) \quad \gamma_s = \begin{cases} q, & p = 1, \\ 2, & p = 2, \\ 3, & p = 3, \\ 4, & p \geq 4. \end{cases}$$

Lemma 2. *When $G' \setminus G'' = \phi$, $S = D \cup X'' \cup Z'$ is MSD of G.*

Proof. Since $G' \setminus G'' = \phi$, we have $G' = G''$. Which imply, the only components of $G \setminus (Z' \cup X'' \cup Y)$ are components isomorphic to P_2, where each vertex has *degree* ≥ 2 in G, components isomorphic to P_4, or components isomorphic to P_7. In all these components, there exists a vertex $v \in D \setminus N[X'] \subset S$ which defends y_1. That is $(S \setminus \{v\}) \cup \{y_1\}$ is a secure dominating set of G. Further for these components $\gamma = \gamma_s$. Which imply D securely dominates all vertices of G''. Also by Lemma 1 and Proposition 2 we observe $S = D \cup X'' \cup Z'$ is MSD of G.

Lemma 3. *When there exists a component C of $G' \setminus G''$ with size $|C| \geq 3$ such that a vertex $x \in X'$ belongs to C, $S = D \cup X'' \cup Y \cup Z'$ is MSD of G.*

Proof. Since there exists a component C of G' with size $|C| \geq 3$ such that a vertex $x \in X'$ belongs to C, we have vertex u which is adjacent to $N[x]$. For such a vertex $u \notin S$ a vertex $y_1 \in S$ is the only vertex which could defend u. That is for $u \in C \setminus S$ there exits $y_1 \in S$ such that $(S \setminus \{y_1\}) \cup \{u\}$ is a secure dominating set of G. For all other vertex $v \in G' \setminus S$ there exists a vertex $v' \in (D \cap N(v)) \subset S$ such that $(S \setminus \{v'\}) \cup \{v\}$ is a secure dominating set of G. Also by Lemma 1 and Proposition 2 we observe $S = D \cup X'' \cup Y \cup Z'$ is MSD of G.

Lemma 4. *When $G' \setminus G'' \neq \phi$ and every component C of $G' \setminus G''$ with size $|C| \geq 3$, has no vertex of X', $S = D \cup X'' \cup Y \cup (Z' \setminus \{z\})$ is MSD of G.*

Proof. Clearly, for $x \in X'$ we have vertex $u \in S$ adjacent to x such that $(S \setminus \{u\}) \cup \{x\}$ is a secure dominating set of G. For all other vertex $v \in G' \setminus S$ there exists a vertex $v' \in (D \cap N(v)) \subset S$ such that $(S \setminus \{v'\}) \cup \{v\}$ is a secure dominating set of G. This holds as $\{y_1\} \in S$ dominates all vertices of $N(v')$. For $z \in Z'$ we have $y_1 \in S$, and $(S \setminus \{y_1\}) \cup \{z\}$ is a secure dominating set of G. By Lemma 1 and Proposition 2 we observe $S = D \cup X'' \cup Y \cup (Z' \setminus \{z\})$ is MSD of G.

We present an algorithm to compute the minimum secure dominating set on chordal bisplit graphs.

Algorithm 1 MSD: Chordal bisplit graphs

Input: A connected chordal bisplit graph $G = (X, Y, Z, E)$.
Output: A minimum secure dominating set S of G.

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Let S' = D ∪ X''.
if G' \ G'' = φ. then
    S = S' ∪ Z'
    return S.
else
    if x ∈ X' belongs to C and |C| ≥ 3 then
        S = S' ∪ Y ∪ Z'
        return S
    else
        S = S' ∪ Y ∪ (Z' \ {z})
        return S
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Theorem 3. *Finding a MSD on chordal bisplit graphs is polynomial time solvable.*

Proof. The result follows from the polynomial time solvability of minimum domination set problem on trees [17].

3.2 Chordal Bipartite Bisplit Graphs

We observed that secure domination problem was polynomial time solvable on chordal bisplit graphs. This motivates us to analyse its complexity on bisplit graphs having a cycle of length four, which are precisely the chordal bipartite bisplit graphs. In this section, we show that *SDD* on chordal bipartite bisplit graphs is NP-hard by presenting a polynomial time reduction. The candidate problem for reduction is *SDD* on *chordal bipartite graphs* which is known to be NP-complete[10].

Construction: Let $G = (X, Y, E)$ be a chordal bipartite graph with bipartitions X and Y. Let vertex set $X = \{k_1, k_2, \ldots, k_p\}$ and $Y = \{l_1, l_2, \ldots, l_q\}$. We construct $G' = (X_1 \cup X_2 \cup Y_1 \cup Y_2, E')$ from G as follows. The vertex set $V(G')$ consists of original vertices of G, a copy of vertices of G labelled as $\{m_1, m_2, \ldots, m_p;\ n_1, n_2, \ldots, n_q\}$ and four additional vertices x, y, m, n. Define $X_1 = \{x, k_i \mid k_i \in X; 1 \le i \le p\}$; $Y_1 = \{y, l_i \mid l_i \in Y; 1 \le i \le q\}$; $X_2 = \{m, m_i \mid k_i \in X; 1 \le i \le p\}$ and $Y_2 = \{n, n_i \mid l_i \in Y; 1 \le i \le q\}$. The edge set E' is defined as $E' = E \cup \{\{m_i, n_j\} \mid \{k_i, l_j\} \in E; 1 \le i \le p, 1 \le j \le q\} \cup \{\{k_i, m_j\} \mid \forall i, j; 1 \le i \le p, 1 \le j \le q\} \cup E^*$ where $E^* = \{\{y, x\}, \{x, m\}, \{m, n\}, \{x, m_i\}, \{m, k_i\} \mid \forall i; 1 \le i \le p\}$. We note that, $|V(G')| = 2 \cdot |V(G)| + 4$ and $|E'| = 2 \cdot |E(G)| + (p+1)^2 + 2$. Thus G' can be constructed in polynomial time. For an illustration of this construction, we refer to Figure 1

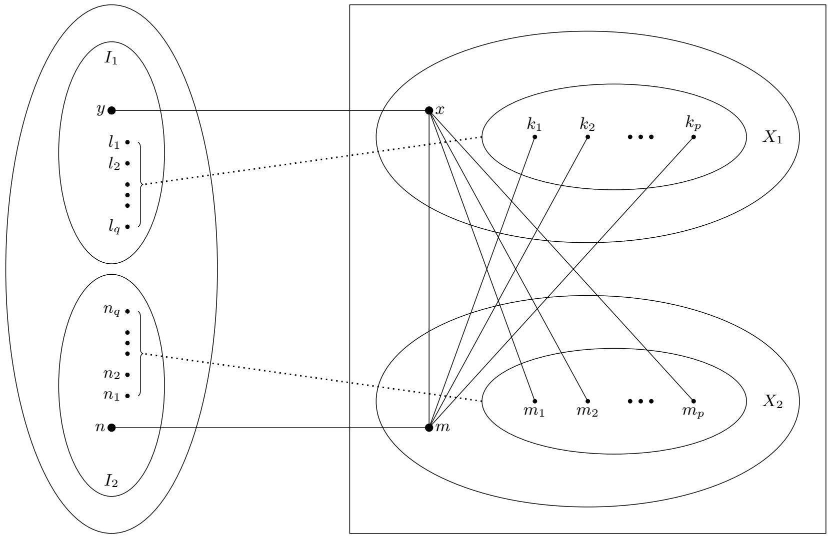

Fig. 1: An illustration of the construction of G' from G

Claim 3.3. G' is a bipartite bisplit graph.

Proof. We first show that $G' = (X', Y', Z', E')$ is a bisplit graph where X', Y' and Z' are three stable sets and $Y' \cup Z'$ forms a biclique. By our construction, $X_1 \cup X_2$ induces a biclique. Also, no two vertices of $Y_1 \cup Y_2$ are adjacent, hence it induces an independent set. Therefore, the graph $G' = (X', Y', Z', E')$ where $X' = Y_1 \cup Y_2$, $Y' = X_1$ and $Z' = X_2$ form a bisplit graph. Further, by our construction vertices of Y_1 are non adjacent to vertices of X_2 and vertices of Y_2 are non adjacent to vertices of X_1. Thus $A = X_1 \cup Y_2$ and $B = X_2 \cup Y_1$ form a bipartition for G'.

Claim 3.4. G' is chordal bipartite.

Proof. Since G is chordal bipartite graph, the subgraphs induced by $X_1 \cup Y_1$ and $X_2 \cup Y_2$ are chordal bipartite. So, to prove G' is chordal bipartite, it suffices to show that every cycle $\mathscr{C}$ of length six involving vertices of both $X_1 \cup Y_1$ and $X_2 \cup Y_2$ has four vertices of biclique forming a C_4. Assume that $\mathscr{C}$ has fewer than four vertices from biclique. Let $\mathscr{C}$ contain three vertices from the biclique $X_1 \cup X_2$. These three vertices induce a P_3 in G'. By our construction, if the pendants of P_3 are in X_2 then the other three vertices of $\mathscr{C}$ are from Y_2 and if the pendants of P_3 are in X_1 then the other three vertices of $\mathscr{C}$ are from Y_1. Also, by Claim 3.3 $G' = A \cup B$ is bipartite. So, such a cycle $\mathscr{C}$ involving four vertices from one of the partitions and two vertices from other is not possible. By similar reasoning we may show that $\mathscr{C}$ cannot contain fewer than three vertices from the biclique. Thus the assumption fails and consequently the claim follows.

Claim 3.5. G has a secure dominating set S with $|S| \leq k$ if and only if G' has a secure dominating set S' with $|S'| \leq 2k + 2$.

Proof. Let G have a secure dominating set S of size at most k. We shall prove that G' has a secure dominating set S' with $|S'| \leq 2k+2$. Define $G^* = G' \setminus \{x, y, m, n\}$. Consider the set $S^* = S_1 \cup S_2$ where $S_1 = \{k_i, l_j \mid k_i \in S, l_j \in S; 1 \leq i \leq p, 1 \leq j \leq q\}$ and $S_2 = \{m_i, n_j \mid k_i \in S, l_j \in S; 1 \leq i \leq p, 1 \leq j \leq q\}$. Let $S' = S^* \cup \{x, m\}$ and $|S'| \leq 2k + 2$. Clearly S' is a dominating set of G' as S^* dominates all vertices of G^* and $\{x, m\} \in S'$ dominates $\{x, y, m, n\}$. Now we need to show S' is a secure dominating set of G'. That is, we will prove for each vertex $v \in V(G') \setminus S'$, there exists a neighbour $v' \in S'$ such that the swap set $(S' \setminus \{v'\}) \cup \{v\}$ is again a dominating set of G'.

- When $v = y$ there exists a neighbour $x \in S'$ such that $(S' \setminus \{x\}) \cup \{y\}$ is a dominating set of G'. This holds as S^*, a subset of the above set dominates all vertices of G^* and $\{y, m\}$ in the above set dominates the four other vertices.
- When $v = n$ there exists a neighbour $m \in S'$ such that $(S' \setminus \{m\}) \cup \{n\}$ is a dominating set of G'. This holds as S^*, a subset of the above set dominates all vertices of G^* and $\{x, n\}$ in the above set dominates the four other vertices.
- When $v \in V(G^*) \setminus S^*$. Since S is a secure dominating set of G, S_1 securely dominates vertices of Y_1 as adjacencies in G are preserved in G'. As a consequence, S^* securely dominates vertices of $Y_1 \cup Y_2$. It suffices to show that, $v \in (V(G^*) \setminus S^*) \cap (X_1 \cup X_2)$ is securely dominated. Since S^* is a subset of S' and S^* is a dominating set of $X_1 \cup X_2$, for every vertex $v \in (V(G^*) \setminus S^*) \cap (X_1 \cup X_2)$, there exists a neighbour $v' \in S^* \subseteq S'$. Therefore, for each $v \in (V(G^*) \setminus S^*) \cap (X_1 \cup X_2)$, there exists a neighbour $v' \in S'$ such that $(S' \setminus \{v'\}) \cup \{v\}$ is a dominating set of G'. This follows as $\{x, m\}$ is a subset of the above set and it dominates all vertices of $X_1 \cup X_2$.

Therefore, $S' = S^* \cup \{x, m\}$ is a secure dominating set of G' with $|S'| \leq 2k + 2$. Conversely, Let G' have a secure dominating set S' with $|S'| \leq 2k + 2$. We shall prove that G has a secure dominating set S with $|S| \leq k$. We note that, since y and n are pendant vertices of G', $|S' \cap \{x, y\}| \geq 1$ and $|S' \cap \{m, n\}| \geq 1$. Let $G^* = G' \setminus \{x, y, m, n\}$. Consider $S^* = S' \cap V(G^*)$. $|S^*| = |S'| - [|S' \cap \{x, y\}| + |S' \cap \{m, n\}|] \leq (2k + 2) - 2 = 2k$. If S^* securely dominates G^*, then we are done. Because, $S = S^* \cap (X_1 \cup Y_1)$ forms a secure dominating set of G and since G^* is symmetric $|S| = |S^*|/2 \leq k$. So, we assume S^* is not a secure dominating set of G^*. Let $W \subseteq V(G^*)$ be the set of vertices that are not securely dominated by S^*. But, W was securely dominated by S' in G'. This implies that W was securely dominated by vertices of $S' \cap \{x, y, m, n\}$ in G^*. Thus, by construction, $W \subseteq X_1 \cup X_2$. Let $W_1 = W \cap X_1$ and $W_2 = W \cap X_2$. Without loss of generality, assume $W_1 \neq \emptyset$. We know that W_1 is securely dominated by S' in G', which implies $m \in S'$. By Proposition 1, $epn(m, S')$ is complete. It follows that W_1 induces a complete subgraph of G' and $n \in S'$. Consequently, $|S' \cap \{m, n\}| = 2$. Further, since X_1 is an independent set, we have $|W_1| = 1$. Now, $W_2 \neq \emptyset$ or $W_2 = \emptyset$. If $W_2 \neq \emptyset$, by similar argument it can be shown that $|W_2| = 1$ and $|S' \cap \{x, y\}| = 2$. If $W_2 = \emptyset$, then $|W_2| = 0$ and $|S' \cap \{x, y\}| \geq 1$. From the above, $|W| = |W_1| + |W_2| \leq |S' \cap \{x, y\}| + |S' \cap \{m, n\}| - 2$. It follows that $|S^*| = |S'| - [|S' \cap \{x, y\}| + |S' \cap \{m, n\}|] \leq |S'| - [|W| + 2]$. Also, $S^* \cup W$ is a secure dominating set of G^* and $|S^* \cup W| = |S^*| + |W| \leq |S'| - [|W| + 2] + |W| =$

$|S'| - 2 \leq 2k$. Further, since G^* is symmetric $S = (S^* \cup W) \cap (X_1 \cup Y_1)$ forms a secure dominating set of G and $|S| = |S^* \cup W|/2 \leq k$.

Theorem 4. *SDD is NP-complete for chordal bipartite bisplit graphs.*

Proof. The proof of Theorem 4 follows from Claims 3.3, 3.4, and 3.5.

We proved that Secure domination problem is NP-Hard on chordal bipartite bisplit graphs. The candidate problem for the reduction is Secure domination problem on chordal bipartite graphs. This indicates that the problem remains computationally intractable even in the presence of structural restrictions such as bounded cycle length. A closer inspection of the reduction instances, however, reveals the presence of arbitrarily long induced paths. This observation naturally leads to the following question. Does bounding the length of induced paths reduce the computational complexity of the secure domination problem? Motivated by this, we initiate a systematic study of the problem under path-length restrictions. As a first step in this direction, we consider the class of P_5-$free$ *chordal bipartite graphs*, also known as chain graphs.

3.3 Chain graphs

In this section, we present an algorithm to compute the minimum secure dominating set on chain graphs. A bipartite graph $G = (X, Y, E)$ is a chain graph, if there exists a chain ordering of $X \cup Y$, i.e. $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ such that $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_n)$ and $N(y_1) \supseteq N(y_2) \supseteq \cdots \supseteq N(y_m)$. The chain ordering of a chain graph can be computed in linear-time [18]. Define a relation R on X as follows. Let x_i and x_j are related if $N(x_i) = N(x_j)$. Observe that R is an equivalence relation. Assume that $X_1, X_2, \ldots, X_k$ is the partition of X based on the relation R. Define $Y_1 = N(X_1)$ and $Y_i = N(X_i) \setminus \bigcup_{j=1}^{i-1} N(X_j)$; $i = 2, 3, \ldots k$. Then, $Y_1, Y_2, \ldots, Y_k$ forms a partition of Y. Such partition $X_1, X_2, \ldots, X_k, Y_1, Y_2, \ldots, Y_k$ of $X \cup Y$ is called a *proper ordered chain partition* of $X \cup Y$. Note that the number of sets in the partition of X and Y is same. Further, the set of pendant vertices of G are contained in the set $X_1 \cup Y_k$. Let $X_1' \subseteq X_1$ and $Y_k' \subseteq Y_k$ denote the set of pendant vertices in G.
Define

$$X_s = \begin{cases} X_1' \setminus \{x_1\} \ , & X_1' \neq \phi, \\ \phi \ , & X_1' = \phi. \end{cases} \qquad Y_s = \begin{cases} Y_k' \setminus \{y_m\} \ , & \mathrm{Y}_k' \neq \phi, \\ \phi \ , & \mathrm{Y}_k' = \phi. \end{cases}$$

Let $S' = \{x_n, y_1\}$ and $S'' = \{x_{n-1}, y_2\}$.

Theorem 5. *Let $G = (X, Y, E)$ be a connected chain graph with chain partition $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$. Let S be a minimum secure dominating set of G. Then S is given by one of the following cases:*

(a) If $|X \setminus X_1'| > 2$ and $|Y \setminus Y_k'| > 2$, then $S = S' \cup S'' \cup X_s \cup Y_s$.
(b) If $|X \setminus X_1'| = 2$, then $S = S' \cup \{x_{n-1}\} \cup X_s \cup Y_s$.

(c) If $|Y \setminus Y'_k| = 2$*, then* $S = S' \cup \{y_2\} \cup X_s \cup Y_s$*.*

(d) If $|X \setminus X'_1| = 1$ *and* $|Y \setminus Y'_k| = 1$*, then* $S = S' \cup X_s \cup Y_s$*.*

(e) If $|X \setminus X'_1| = 0$ *or* $|Y \setminus Y'_k| = 0$*, then* $S = (X \cup Y) \setminus \{x_1\}$

Proof. Observe that y_1 is adjacent to all vertices of X and y_2 is adjacent to all vertices of $X \setminus X'_1$. Similarly, x_n is adjacent to all vertices of Y and x_{n-1} is adjacent to all vertices of $Y \setminus Y'_k$. This holds due to chain ordering of G. Note that $N(X'_1) = \{y_1\}$ and $N(Y'_k) = \{x_n\}$.

(a) Let $|X \setminus X'_1| > 2$ and $|Y \setminus Y'_k| > 2$. Note that $\{x_n, x_{n-1}\} \cup (Y \setminus Y'_k)$ and $\{y_1, y_2\} \cup (X \setminus X'_1)$ are isomorphic to $K_{2,q}$. Further $\{x_n\} \cup Y'_k$ and $\{y_1\} \cup X'_1$ are isomorphic to $K_{1,q}$. Therefore by Proposition 2 $S = S' \cup S'' \cup X_s \cup Y_s$ is securely dominates G. It remains to prove S is optimal. Clearly, X_s and Y_s are the minimal possible number of leaves included in S. Observe $\{y_1\}$ is the only vertex which defends $\{x_1\}$, and $\{x_n\}$ is the only vertex that defends $\{y_m\}$. Since $|X \setminus X'_1| > 2$, for a non-pendant vertex $v \in (X \setminus S)$, there exists $y_2 \in S$ such that $(S \setminus \{y_2\}) \cup \{v\}$ is a dominating set of G. The necessity of $x_{n-1} \in S$ follows by similar reasoning. Therefore, $S = S' \cup S'' \cup X_s \cup Y_s$ is MSD of G in this case.

(b) Since $|X \setminus X'_1| = 2$, there is no vertex in $X \setminus X'_1$ which is uniquely defended by y_2. Hence we omit y_2 in MSD of G from previous case. Therefore, $S = S' \cup \{x_{n-1}\} \cup X_s \cup Y_s$ is MSD of G in this case.

(c) The proof of this case is analogous to that of the previous case

(d) Note that $|X \setminus X'_1| = 1$ imply $|Y \setminus Y'_k| = 1$. By Proposition 2, it is immediate that $S = S' \cup X_s \cup Y_s$ is a MSD of G.

(e) Let $|X \setminus X'_1| = 0$ or $|Y \setminus Y'_k| = 0$ This imply that $G = K_{1,q}$. By Proposition 2, it follows that at most one vertex of G can be excluded from a minimal secure dominating set. Therefore $S = (X \cup Y) \setminus \{x_1\}$ is MSD of G in this case.

We present an algorithm to compute the minimum secure dominating set on chain graphs.

Algorithm 2 MSD: Chain graphs

Input: A connected chain graph $G = (X \cup Y, E)$ with proper order chain partition $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$ of X and Y.
Output: A minimum secure dominating set S of G.

```
if |X \ X'_1| > 2 and |Y \ Y'_k| > 2 then
    S = S' ∪ S'' ∪ X_s ∪ Y_s
    return S.
else
    if |X \ X'_1| = 2 then
        S = S' ∪ {x_{n-1}} ∪ X_s ∪ Y_s
        return S.
    else
        if |Y \ Y'_k| = 2 then
            S = S' ∪ {y_2} ∪ X_s ∪ Y_s
            return S.
        else
            if |X \ X'_1| = 1 and |Y \ Y'_k| = 1 then
                S = S' ∪ X_s ∪ Y_s
                return S.
            else
                |X \ X'_1| = 0 or |Y \ Y'_k| = 0
                S = (X ∪ Y) \ {y_1}
                return S.
```

Theorem 6. *Finding a MSD on chain graphs is polynomial time solvable.*

Proof. The chain ordering of a chain graph can be computed in linear-time [18]. Further, optimality follows from Theorem 5. Therefore, by Algorithm 2, MSD is polynomial time solvable on chain graphs.

4 Approximability of Secure Domination Problem

4.1 Inapproximability of Secure Domination Problem in Bisplit Graphs

We investigate the approximation hardness of Secure domination problem in bisplit graphs. To achieve this, we need the following result on MSD in split graphs.

Theorem 7. *[19] If there is some $\epsilon > 0$ such that a polynomial time algorithm can approximate the secure domination problem for a split graph $G = (V, E)$ within a ratio of $(1 - \epsilon)$ ln $|V|$, then $NP \subseteq DTIME(|V|^{O(log\ log\ |V|)})$.*

By using theorem 7, we will prove similar result for MSD in bisplit graphs.

Theorem 8. *If there is some $\epsilon > 0$ such that a polynomial time algorithm can approximate the secure domination problem for a bisplit graph $G = (V, E)$ within a ratio of $(1 - \epsilon)$ ln $|V|$, then $NP \subseteq DTIME(|V|^{O(log\ log\ |V|)})$.*

Proof. We establish an approximation preserving reduction from SDD in split graphs to SDD in bisplit graphs. Let $G = (K, I, E)$ be a split graph where $K = \{k_1, k_2, \ldots, k_p\}$ induces a clique and $I = \{l_1, l_2, \ldots, l_q\}$ induces an independent set. We construct $G' = (K_1 \cup K_2 \cup I_1 \cup I_2, E')$ from G as follows. The vertex set consists of original vertices of G, a copy of vertices of G labelled as $\{m_1, m_2, \ldots, m_p; n_1, n_2, \ldots, n_q\}$ and four additional vertices x, y, m, n. Define $K_1 = \{x, k_i \mid k_i \in K; 1 \leq i \leq p\}$; $I_1 = \{y, l_i \mid l_i \in I; 1 \leq i \leq q\}$; $K_2 = \{m, m_i \mid k_i \in K; 1 \leq i \leq p\}$ and $I_2 = \{n, n_i \mid l_i \in I; 1 \leq i \leq q\}$. The edge set E' is modified by removing $E(K)$ and adding additional edges. Define $E' = (E(G) \setminus E(K)) \cup \{\{m_i, n_j\} \mid \{k_i, l_j\} \in E; 1 \leq i \leq p, 1 \leq j \leq q\} \cup \{\{k_i, m_j\} \mid \forall i, j; 1 \leq i, j \leq p\} \cup E^*$ where $E^* = \{\{y, x\}, \{x, m\}, \{m, n\}, \{x, m_i\}, \{m, k_i\} \mid \forall i; 1 \leq i \leq p\}$. We note that $|V'| = 2 \cdot |V(G)| + 4$, $|E'| = 2 \cdot |E(G) \setminus E(K)| + (p + 1)^2 + 2$.Thus G' can be constructed in polynomial time.

Claim 4.1. G' is a bisplit graph.

Proof. We show that $G' = (X, Y, Z, E')$ is a bisplit graph where X, Y and Z are three stable sets and $Y \cup Z$ forms a biclique. By our construction, $K_1 \cup K_2$ induces a biclique. Also, no two vertices of $I_1 \cup I_2$ are adjacent; hence it induces an independent set. Therefore, the graph $G' = (X, Y, Z, E')$ where $X = I_1 \cup I_2$, $Y = K_1$ and $Z = K_2$ forms a bisplit graph.

Claim 4.2. G has a secure dominating set S with $|S| \leq k$ if and only if G' has a secure dominating set S' with $|S'| \leq 2k + 2$.

Proof. Define $G^* = G' \setminus \{x, y, m, n\}$. Consider the set $S^* = S_1 \cup S_2$ where $S_1 = \{k_i, l_j \mid k_i \in S, l_j \in S; 1 \leq i \leq p, 1 \leq j \leq q\}$ and $S_2 = \{m_i, n_j \mid k_i \in S, l_j \in S; 1 \leq i \leq p, 1 \leq j \leq q\}$. Let $S' = S^* \cup \{x, m\}$ and $|S'| \leq 2k + 2$. Clearly S' is a dominating set of G' as S^* dominates all vertices of G^* and $\{x, m\} \in S'$ dominates $\{x, y, m, n\}$. Now we need to show S' is a secure dominating set of G'. That is, we will prove for each vertex $v \in V(G') \setminus S'$, there exists a neighbour $v' \in S'$ such that the swap set $(S' \setminus \{v'\}) \cup \{v\}$ is again a dominating set of G'.

- When $v = y$ there exists a neighbour $x \in S'$ such that $(S' \setminus \{x\}) \cup \{y\}$ is a dominating set of G'. This holds as S^*, a subset of the above set dominates all vertices of G^* and $\{y, m\}$ in the above set dominates the four other vertices.
- When $v = n$ there exists a neighbour $m \in S'$ such that $(S' \setminus \{m\}) \cup \{n\}$ is a dominating set of G'. This holds as S^*, a subset of the above set dominates all vertices of G^* and $\{x, n\}$ in the above set dominates the four other vertices.
- When $v \in V(G^*) \setminus S^*$. Since S is a secure dominating set of G, S_1 securely dominates vertices of I_1 as adjacencies across K and I in G are preserved in G'. As a consequence, S^* securely dominates vertices of $I_1 \cup I_2$ and due to removal of edges $E(K)$ in G, we see that S^* dominates vertices of $K_1 \cup K_2$. It suffices to show that, $v \in (V(G^*) \setminus S^*) \cap (K_1 \cup K_2)$ is securely dominated. Since S^* is a subset of S' and S^* is a dominating set of $K_1 \cup K_2$, for every vertex $v \in (V(G^*) \setminus S^*) \cap (K_1 \cup K_2)$, there exists a neighbour $v' \in S^* \subseteq S'$. Therefore, for each $v \in (V(G^*) \setminus S^*) \cap (K_1 \cup K_2)$, there exists a neighbour $v' \in S'$ such that $(S' \setminus \{v'\}) \cup \{v\}$ is a dominating set of G'. This follows as $\{x, m\}$ is a subset of the above set and it dominates all vertices of $K_1 \cup K_2$.

Therefore, $S' = S^* \cup \{k, m\}$ is a secure dominating set of G' with $|S'| \leq 2k + 2$. Conversely, let G' have a secure dominating set S' with $|S'| \leq 2k + 2$. We shall prove that G has a secure dominating set S with $|S| \leq k$. We note that, since y and n are pendant vertices of G', $|S' \cap \{x, y\}| \geq 1$ and $|S' \cap \{m, n\}| \geq 1$. Let $G^* = G' \setminus \{x, y, m, n\}$. Consider $S^* = S' \cap V(G^*)$. $|S^*| = |S'| - [|S' \cap \{x, y\}| + |S' \cap \{m, n\}|] \leq (2k + 2) - 2 = 2k$. If S^* securely dominates G^*, then we are done. Because, $S = S^* \cap (K_1 \cup I_1)$ forms a secure dominating set of G and since G^* is symmetric $|S| = |S^*|/2 \leq k$. So, we assume S^* is not a secure dominating set of G^*. Let $W \subseteq V(G^*)$ be the set of vertices that are not securely dominated by S^*. But, W was securely dominated by S' in G'. This implies that W was securely dominated by vertices of $S' \cap \{x, y, m, n\}$ in G^*. Thus, by construction, $W \subseteq K_1 \cup K_2$.
Let $W_1 = W \cap K_1$ and $W_2 = W \cap K_2$. Without loss of generality, assume $W_1 \neq \emptyset$. We know that W_1 is securely dominated by S' in G', which implies $m \in S'$. By Proposition 1, $epn(m, S')$ is complete. It follows that W_1 induces a complete subgraph of G' and $n \in S'$.Consequently, $|S' \cap \{m, n\}| = 2$. Further, since K_1 is an independent set, we have $|W_1| = 1$. Now, $W_2 \neq \emptyset$ or $W_2 = \emptyset$. If $W_2 \neq \emptyset$, by similar argument it can be shown that $|W_2| = 1$ and $|S' \cap \{x, y\}| = 2$. If $W_2 = \emptyset$, then $|W_2| = 0$ and $|S' \cap \{x, y\}| \geq 1$.
From the above, $|W| = |W_1| + |W_2| \leq |S' \cap \{x, y\}| + |S' \cap \{m, n\}| - 2$. It follows that $|S^*| = |S'| - [|S' \cap \{x, y\}| + |S' \cap \{m, n\}|] \leq |S'| - [|W| + 2]$. Also, $S^* \cup W$ is a secure dominating set of G^* and $|S^* \cup W| = |S^*| + |W| \leq |S'| - [|W| + 2] + |W| = |S'| - 2 \leq 2k$. Further, since G^* is symmetric $S = (S^* \cup W) \cap (K_1 \cup I_1)$ forms a secure dominating set of G and $|S| = |S^* \cup W|/2 \leq k$.

Let us assume that there exists some (fixed) $\epsilon > 0$ such that MSD for bisplit graphs with $|V'|$ vertices can be approximated within a ratio of $\alpha = (1 - \epsilon)\ ln\ |V'|$ by a polynomial time algorithm $\mathcal{A}$. Let $x > 0$ be a fixed integer with $x > \frac{1}{\epsilon}$. By using algorithm $\mathcal{A}$, we construct a polynomial time algorithm for MSD in split graphs as follows.

Algorithm 3 Approx-MSD split

Input: A Split graph $G = (K, I, E)$
Output: A minimum secure dominating set S of G
if there exists a minimum secure dominating set S of G with $|S| < x$ **then**
  return S
else
  Construct the bisplit graph G' as described above
  Compute a MSD S' in G^* using algorithm $\mathcal{A}$
  if $W = \emptyset$ or $epn_{G^*}(m, S') = \emptyset$ **then**
    $S \leftarrow S' \cap V$
    return S
  if $W \neq \emptyset$ and $epn_{G^*}(m, S') \neq \emptyset$ **then**
    $S \leftarrow (S' \cap V) \cup \{v\}$ for some $v \in epn_{G^*}(m, S')$
    return S

Initially, if there is a minimum secure dominating set S of G with $|S| < x$, then it can be computed in polynomial time. Now, since the algorithm $\mathcal{A}$ runs in polynomial time, the Algorithm 3 also runs in polynomial time. If the returned set S satisfies $|S| < x$ then S is a minimum secure dominating set of G and we are done. In the following, we will analyse the case when Algorithm 3 returned the set S with $|S| \geq x$.
Let S_0 and S_0' be minimum secure dominating set of G and minimum secure dominating set of G', respectively. By Claim 4.2 we have $|S_0'| = 2 \cdot |S_0| + 2 = 2 \cdot (|S_0| + 1)$, where $|S_0| \geq x$. Now Algorithm 3 can compute a secure dominating set of G of size $|S| \leq \frac{|S'|-2}{2} = \frac{|S'|}{2} - 1 \leq \frac{\alpha \cdot |S_0'|}{2} - 1 = \frac{\alpha \cdot 2 \cdot (|S_0|+1)}{2} - 1 = \alpha \cdot (|S_0| + 1) - 1 < \alpha \cdot (1 + \frac{1}{|S_0|})|S_0|$. Since $|S_0| \geq x$, $|S| < \alpha \cdot (1 + \frac{1}{x})|S_0|$. Hence, Algorithm 3 approximates secure domination problem for given split graph G within the ratio $\alpha \cdot (1 + \frac{1}{x})$. Also x is a positive integer such that $x > \frac{1}{\epsilon}$. It follows that $\alpha \cdot (1 + \frac{1}{x}) \leq \alpha \cdot (1 + \epsilon) = (1 - \epsilon) \cdot (1 + \epsilon)\ ln\ |V'| = (1 - \epsilon')\ ln\ |V|$, where $\epsilon' = \epsilon^2$. Also, $ln\ |V'| = ln\ (2 \cdot |V| + 4) \approx\ ln\ |V|$ for sufficiently large value of $|V|$. Therefore, Algorithm 3 approximates secure domination problem in split graphs within a ratio of $(1 - \epsilon)\ ln\ |V|$ for some $\epsilon > 0$. This contradiction to Theorem 7 completes the proof.

4.2 Approximation Algorithms for General Graphs

Before we present an approximation algorithm for minimum secure domination problem (MSD) we will recall some definitions and results that are important for our investigation. We begin with the concept of 2-domination and double domination. Let $G = (V, E)$ be a simple graph. A subset $S \subseteq V$ is a $2-$ *dominating set* if every vertex of $V \setminus S$ has at least two neighbours in S. The 2-domination number $\gamma_2(G)$ represent the cardinality of a minimum 2-dominating set of G. Given a graph $G = (V, E)$, in minimum 2-dominating set problem ($M2D$), it is required to find a 2-dominating set $S \subseteq V$ of minimum cardinality. A subset $S \subseteq V$ is a *double dominating set* if S is a 2-dominating set and the subgraph induced by S has no isolated vertex. The double domination number $\gamma_{\times 2}(G)$ represent the cardinality of a minimum double dominating set of G. Given a graph $G = (V, E)$, in minimum double dominating set problem (MDD), it is required to find a double dominating set $S \subseteq V$ of minimum cardinality.
To obtain the approximation ratio of MSD, we require the approximation ratio of the minimum 2-dominating set problem ($M2D$).

Theorem 9. *[20] $M2D$ can be approximated with an approximation ratio of $O(ln\ |V|)$, where V is the vertex set of the input graph G. It can also be approximated within a factor of $1 + ln(\Delta + 2)$, where Δ is the maximum degree of G.*

Observation: A 2-dominating set of a graph G is also a secure dominating set of G.
In [6], Merouane and Chellali established that $\gamma_2(G) \leq \gamma_{\times 2}(G) \leq 2 \cdot \gamma_s(G)$ for a graph G without isolated vertices. Further, by the above observation, the

approximation algorithm described in Theorem 9 also returns a secure domination set of G. Therefore, we propose the following approximation algorithm for MSD whose approximation ratio is a logarithmic factor of the number of vertices of input graph.

Algorithm 4 Approx-MSD

Input: A graph $G = (V, E)$.
Output: A minimum secure dominating set S of G.
Compute a 2-dominating set S' of G (as described in Theorem 9)
$S = S'$.
return S

Theorem 10. *MSD can be approximated with an approximation ratio of $O(ln\ |V|)$, for graphs with $\delta(G) \geq 2$. It can also be approximated within a factor of $2 + 2 \cdot ln(\Delta + 2)$, where Δ is the maximum degree of G.*

Proof. Let $G = (V, E)$ be a connected graph with $\delta(G) \geq 2$. Let S' be a 2-dominating set of G. For every vertex $v \in V \setminus S'$ there exists a vertex $u \in S'$ such that $D = (S' \setminus \{u\}) \cup \{v\}$ is a dominating set of G. Suppose not, then there exists a vertex $x \in V \setminus D$ such that no vertex of D dominates x. Which implies that $N(x) \cap S' = \{u\}$. This contradicts the fact that S' is a 2-dominating set of G. Therefore, every 2-dominating set of a graph G is also a secure dominating set of G.
Let S be the secure dominating set of G computed by the Algorithm 4. By Theorem 9, we have $|S| \leq O(ln|V|) \cdot \gamma_2(G)$. Also, by above observation, we have $|S| \leq O(ln|V|) \cdot \gamma_2(G) \leq 2 \cdot O(ln|V|) \cdot \gamma_s(G) = O(ln|V|) \cdot \gamma_s(G)$. Similarly, it can be observed that $|S| \leq [2 + 2 \cdot ln(\Delta + 2)]\gamma_s(G)$.

5 Conclusion

We investigated the complexity of *Secure Domination Problem* and proved that the problem is NP-complete on bisplit graphs. Having found that SDD on bisplit graphs is NP-complete, our next focus was to analyse the complexity of SDD by restricting the cycle length. We observed that SDD was polynomial time solvable on chordal bisplit graphs and it is the presence of cycles of length four which makes the problem hard on bisplit graphs, thereby delineating the boundary between tractable and intractable cases. Also MSD is polynomial time solvable in chain graphs. Apart from these we have also established hardness and approximation results for secure domination problem.

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